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Self-Diagonal Tensor Powers of Quantum Groups and R-Matrices for Tensor Products of Representations

Ralf A. Engeldinger[1]

Sektion Physik der Ludwig-Maximilians-Universität München
Lehrstuhl Professor Wess
Theresienstraße 37, D-80333 München, Federal Republic of Germany

e-mail: engeldin@lswes8.ls-wess.physik.uni-muenchen.de

Abstract

Twisted tensor powers of quasitriangular Hopf algebras with diagonal sub-Hopf-algebras (self-diagonal tensor powers) are introduced together with their duals and their mutual ∗-structures as generalizations of the Drinfel'd double as given by Reshetikhin and Semenov-Tian-Shansky. R-Matrices for tensor products of representations are derived.

1 Introduction

In [CEJSZ] the Drinfel'd double of a quasitriangular Hopf algebra (QTHA) $(\mathcal{U}, \mathcal{R})$ as a twisted tensor product of two copies of $\mathcal{U}$ (originally given by Reshetikhin and Semenov-Tian-Shansky [ReSe]) was recovered as the result of an attempt to define a tensor product of two copies of $\mathcal{U}$ in such a way that the image of the coproduct of $\mathcal{U}$ is a (so-called diagonal) sub-Hopf algebra (which for the standard tensor product Hopf algebra is not the case).

Tensor products of representations of Hopf algebras are defined via the coproduct. This way one immediately obtains R-matrices for tensor products of representations of quasitriangular Hopf algebras from the universal R-matrix. But the R-matrices obtained this way are not the only possible ones. Lorek, Schmidke and Wess [LSW] constructed all R-matrices for the $[3] \oplus [1]$–representation of $U_q su(2)$: the standard $SO_{q^2}(3)$-R-matrix and the two $SO_q(1,3)$-R-matrices. The former is the one obtainable by applying the coproduct on the universal R-matrix of $U_q su(2)$. The latter ones, on the other hand, were shown in [CEJSZ] to naturally arise from the universal R-matrices of the twisted tensor product of two copies of $U_q su(2)$. It was this observation that inspired the present work.

Explicitly it looks as follows. Using the product form of the representation

$$[2] \times [2] = [3] \oplus [1]$$

and the notation

$$\langle [i] \times [j], h \rangle = \langle [i] \otimes [j], \Delta(h) \rangle$$

we can write

$$\langle ([2] \times [2]) \otimes ([2] \times [2]), \mathcal{R} \rangle = R_{SO(3)} \oplus 1_{[3] \otimes [1]} \oplus 1_{[1] \otimes [3]} \oplus 1_{[1] \otimes [1]} \tag{1}$$

$$\langle ([2] \otimes [2]) \otimes ([2] \otimes [2]), \mathcal{R}_{41}^{-1} \mathcal{R}_{13} \mathcal{R}_{42}^{-1} \mathcal{R}_{23} \rangle = R_{I_{SO(1,3)}} \tag{2}$$

$$\langle ([2] \otimes [2]) \otimes ([2] \otimes [2]), \mathcal{R}_{41}^{-1} \mathcal{R}_{31}^{-1} \mathcal{R}_{42}^{-1} \mathcal{R}_{23} \rangle = R_{II_{SO(1,3)}} \tag{3}$$

In this paper the same is done for the tensor product of an arbitrary number s of copies of $\mathcal{U}$. Like in [CEJSZ] for the case $s = 2$ the corresponding Hopf dual (for $s = 2$ also discussed in [Pod]) and the respective $*$-structures are given. Hence self-diagonal tensor power Hopf algebras can be regarded as generalizations of the Drinfel'd double or of complex quantum groups [DSWZ]. Following Wess' idea we then derive R-matrices for tensor products of representations from the universal R-matrices of quasitriangular self-diagonal tensor power Hopf algebras.

2   Self-diagonal Tensor Power Hopf Algebras

The tensor product of two copies of a quasitriangular Hopf algebra $(\mathcal{U}, \mathcal{R})$ is again a QTHA in a natural way. The same is true for the tensor product of an arbitrary number of such copies.

To number tensor factors of $\mathcal{U}^{\otimes s} \otimes \mathcal{U}^{\otimes s}$ we use both a natural notation

$$(1_1, 1_2, \ldots, 1_s, 2_1, \ldots, 2_s)$$

and a 'flattened' one

$$(1, 2, \ldots, s, s+1, \ldots, 2s),$$

and additionally

$$\bar{1} = (1, \ldots, s) = (1_1, \ldots, 1_s), \qquad \bar{2} = (s+1, \ldots, 2s) = (2_1, \ldots, 2_s).$$

$$\mathrm{T}(a \otimes b) = b \otimes a$$

$$\mathrm{T}_i = \mathrm{id}^{\otimes i-1} \otimes \mathrm{T} \otimes \mathrm{id}^{\otimes 2s-i-1}$$

$$\mathrm{T}_\Delta^{(s)} = \overset{s-2}{\underset{i=0}{\bigcirc}} \left(\overset{i}{\underset{j=0}{\bigcirc}} \mathrm{T}_{s-i+2j} \right), \qquad \mathrm{T}_m^{(s)} = \mathrm{T}_\Delta^{(s)-1} = \overset{0}{\underset{i=s-2}{\bigcirc}} \left(\overset{i}{\underset{j=0}{\bigcirc}} \mathrm{T}_{s-i+2j} \right). \tag{4}$$

$\bigcirc$ denotes iterated concatenation of mappings ($\circ$).

Then the standard tensor power QTHA has the following form:

$$\begin{aligned}
&\bar{\mathcal{U}} \equiv \mathcal{U}^{\otimes s}, \quad \bar{m} = m^{\otimes s} \circ \mathrm{T}_m^{(s)}, \quad \bar{\Delta} = \mathrm{T}_\Delta^{(s)} \circ \Delta^{\otimes s}, \quad \bar{\varepsilon} = \varepsilon^{\otimes s}, \quad \bar{S} = S^{\otimes s}, \\
&\bar{\mathcal{R}}_{\bar{1}\bar{2}} = \prod_{i=1}^{s} \mathcal{R}_{i,s+i}^{(i)} = \prod_{i=1}^{s} \mathcal{R}_{1_i 2_i}^{(i)}
\end{aligned} \tag{5}$$

where for any i : $\mathcal{R}_{12}^{(i)} \in \{\mathcal{R}_{12}, \mathcal{R}_{21}^{-1}\}$.

In straightforward generalization of the case $s = 2$ we now introduce a different QTHA structure on the same algebra which has a diagonal sub-Hopf algebra isomorphic to $\mathcal{U}$. It is obtained from the standard tensor power QTHA through a twist [Dri]. We denote this self-diagonal tensor product by $\circledR$. Introducing the following left action on a Hopf algebra:

$$\overrightarrow{z}(x) = z \, x \, z^{-1} \tag{6}$$

and

$$\mathrm{R}_k^{\mathcal{F}} = \mathrm{T}_k \circ \overrightarrow{\mathcal{R}}_{k,k+1}^{\mathcal{F}}, \qquad \mathrm{R}^{(s)} = \overset{s-1}{\underset{i=1}{\bigcirc}} \left(\overset{i}{\underset{j=1}{\bigcirc}} \mathrm{R}_{s-i-1+2j}^{\mathcal{F}} \right) \tag{7}$$

$$\mathcal{F} = \prod_{i=1}^{s-1} \left(\prod_{j=1}^{i} \mathcal{R}_{s+j,s-i+j}^{\mathcal{F}} \right) = \prod_{i=s-1}^{1} \left(\prod_{j=1}^{s-i} \mathcal{R}_{2_j 1_{i+j}}^{\mathcal{F}} \right) \tag{8}$$

$$v = \mathcal{F}^{(\bar{1})} \bar{S}(\mathcal{F}^{(\bar{2})}) = \left[\prod_{i=1}^{s-1} \left(\prod_{j=i+1}^{s} \mathcal{R}_{ij}^{\mathcal{F}} \right) \right]^{-1}, \quad v^{-1} = \bar{S}^2(\mathcal{F}^{(\bar{1})}) \mathcal{F}^{(\bar{2})} \tag{}$$

where $\mathcal{R}_{12}^{\mathcal{F}} \in \{\mathcal{R}_{12}, \mathcal{R}_{21}^{-1}\}$, we define the self-diagonal tensor power QTHA by

$$\boldsymbol{\mathcal{U}} \equiv \mathcal{U}^{\circledR s} = \mathcal{U}^{\otimes s} \text{ as vector space over } \mathbb{C},$$
$$\boldsymbol{m} = \bar{m}, \quad \boldsymbol{\Delta} = \mathrm{R}^{(s)} \circ \Delta^{\otimes s} = \overrightarrow{\mathcal{F}} \circ \bar{\Delta}, \quad \boldsymbol{\varepsilon} = \bar{\varepsilon}, \quad \boldsymbol{S} = \overrightarrow{v} \circ \bar{S}, \tag{9}$$
$$\boldsymbol{\mathcal{R}}_{\bar{1}\bar{2}} = \mathcal{F}_{\bar{2}\bar{1}} \bar{\mathcal{R}}_{\bar{1}\bar{2}} \mathcal{F}_{\bar{1}\bar{2}}^{-1}.$$

Since the definition of the standard tensor power Hopf algebra is self-dual we can obtain the dual Hopf algebra (Hopf dual) of the tensor power QTHA $(\mathcal{U}^{\otimes s}, \bar{\mathcal{R}})$ of a QTHA $(\mathcal{U}, \mathcal{R})$ as the tensor power Hopf algebra $\mathcal{A}^{\otimes s}$ of the latter one's dual $\mathcal{A}$. By dualization of the twist transformation we get the dual of the self-diagonal tensor power QTHA $(\mathcal{U}^{\circledR s}, \boldsymbol{\mathcal{R}})$ which we denote as $\mathcal{A}^{\circledR s}$. (The symbol $\circledR$ has a different meaning here.) Introducing the following right action on the dual Hopf algebra

$$\overleftarrow{z}(a) = \langle a_{(\bar{1})}, z \rangle \, a_{(\bar{2})} \, \langle a_{(\bar{3})}, z^{-1} \rangle \tag{10}$$

we define it to be

$$\boldsymbol{\mathcal{A}} \equiv \mathcal{A}^{\circledR s} = \bar{\mathcal{A}} \text{ as vector space over } \mathbb{C},$$
$$\boldsymbol{m} = \bar{m} \circ \overleftarrow{\mathcal{F}}, \qquad \boldsymbol{\Delta} = \bar{\Delta}, \qquad \boldsymbol{\varepsilon} = \bar{\varepsilon}, \qquad \boldsymbol{S} = \bar{S} \circ \overleftarrow{v}. \tag{11}$$

A tensor power Hopf algebra allows various inequivalent $*$-structures (i.e. not related via (co-)conjugation with an invertible element: $h^\dagger \neq \overrightarrow{z}(h^*)$ or $a^\dagger \neq \overleftarrow{z}(a^*)$). The naive choice is simply $\bar{*} = *^{\otimes s}$. From this we can obtain an inequivalent one for every permutation $\pi \in \mathcal{S}_s$ with $\pi^2 = \mathrm{id} : \bar{*}_\pi = \pi \circ \bar{*}$.

We now focus on the case

$$\mathcal{R}_{12}^{*\otimes*} = \mathcal{R}_{21}. \tag{12}$$

Denoting by ι the inversion in $\mathcal{S}_s$ this implies:

$$\iota(v^{\bar{*}}) = v. \tag{13}$$

Let $u \in \mathcal{A} \otimes \mathbb{C}^{n \times n}$ be a fundamental representation of $\mathcal{U}$ (generating all irreducible ones) such that $R_{12} = \langle u_1 \otimes u_2, \mathcal{R} \rangle$ fulfills

$$\overline{R_{12}^{\top}} = R_{21}. \tag{14}$$

In $\mathcal{A}^{\circledR s}$ we have[2]

$$\langle b^{\textstyle *}{}_{(\bar{1})} \otimes a^{\textstyle *}{}_{(\bar{1})}, \mathcal{F} \rangle \, b^{\textstyle *}{}_{(\bar{2})} a^{\textstyle *}{}_{(\bar{2})} \, \langle b^{\textstyle *}{}_{(\bar{3})} \otimes a^{\textstyle *}{}_{(\bar{3})}, \mathcal{F}^{-1} \rangle = b^{\textstyle *} \bullet a^{\textstyle *}$$

$$= (a \bullet b)^{\textstyle *} = \overline{\langle a_{(\bar{1})} \otimes b_{(\bar{1})}, \mathcal{F} \rangle} \, (a_{(\bar{2})} b_{(\bar{2})})^{\textstyle *} \, \overline{\langle a_{(\bar{3})} \otimes b_{(\bar{3})}, \mathcal{F}^{-1} \rangle}. \tag{15}$$

With

$$F = \prod_{i=1}^{s-1} \left(\prod_{j=1}^{i} R^{\mathcal{F}}_{s+j,\,s-i+j} \right) = \prod_{i=s-1}^{1} \left(\prod_{j=1}^{s-i} R^{\mathcal{F}}_{2_j 1_{i+j}} \right) \tag{16}$$

this implies $\overline{F_{\bar{1}\bar{2}}} = (\iota \otimes \iota)(F_{\bar{1}\bar{2}})$ if we assume $* = \bar{*}$. In general this is not true. However there is a different choice which always works in our specified case:

$$* = \iota \circ \bar{*} \qquad (\text{on } \boldsymbol{\mathcal{A}}). \tag{17}$$

Because of eq.(13) we can define (cf.[CEJSZ])

$$* = \vec{v} \circ \iota \circ \bar{*} \qquad (\text{on } \boldsymbol{\mathcal{U}}). \tag{18}$$

The natural generators of $\mathcal{A}^{\circledR s}$ are

$$u^{(i)} = \mathbb{I}^{\otimes i-1} \otimes u \otimes \mathbb{I}^{\otimes s-i} \tag{19}$$

while those of $\mathcal{U}^{\circledR s}$ are as usual the corresponding semirepresentations of $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{R}}^{-1}$

$$L^{+(i)} = \langle \cdot \otimes u^{(i)}, \boldsymbol{\mathcal{R}} \rangle \tag{20}$$

$$L^{-(i)} = \langle u^{(i)} \otimes \cdot, \boldsymbol{\mathcal{R}}^{-1} \rangle. \tag{21}$$

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[2] $a \bullet b = \boldsymbol{m}(a \otimes b)$

They consist of $s + 1$ different matrices of generators[3] (cf.[CEJSZ]):

$$\ell_1^{(j)} = \bigotimes_{k=1}^{s-j} \ell_1^+ \otimes \bigotimes_{k=1}^{j} \ell_1^-, \qquad j = 0, \ldots, s \qquad \text{if} \quad \mathcal{R}^{\mathcal{F}} = \mathcal{R} \qquad (22)$$

or

$$\ell_1^{(j)} = \bigotimes_{k=1}^{s-j} \ell_1^- \otimes \bigotimes_{k=1}^{j} \ell_1^+, \qquad j = 0, \ldots, s \qquad \text{if} \quad \mathcal{R}^{\mathcal{F}} = \mathcal{R}_{21}^{-1} \qquad (23)$$

where $\quad \ell^+ = \langle \cdot \otimes u, \mathcal{R} \rangle \quad$ and $\quad \ell^- = \langle u \otimes \cdot, \mathcal{R}^{-1} \rangle.$

We close this section with the following observation:

Δ^{s-1} is a $*$-algebra isomorphism[4] and as such canonically gives rise to an isomorphism of quasitriangular $*$-Hopf algebras. Its image is a quasitriangular sub-$*$-Hopf algebra of $\mathcal{U}^{\otimes s}$. Therefore $(\mathcal{U}^{\otimes s}, \mathcal{R})$ is called *quasitriangular self-diagonal tensor power $*$-Hopf algebra.*

3 R-Matrices for Tensor Products of Representations

Tensor products of representations naturally live on the corresponding tensor power of the algebra. This way a coproduct on the algebra endows the category of representations of the algebra with a monoidal structure. With the image of Δ^{s-1} being a sub-$*$-Hopf algebra of $\mathcal{U}^{\otimes s}$ we can take advantage of the fact that the latter one is equipped with quasitriangular structures that are not inherited directly from $(\mathcal{U}, \mathcal{R})$ via the isomorphism. They give rise to R-matrices for tensor products of representations of $(\mathcal{U}, \mathcal{R})$ that cannot be obtained by simply applying $\Delta^{s-1} \otimes \Delta^{s-1}$ on $\mathcal{R}$ or $\mathcal{R}_{21}^{-1}$. To the latter ones we will refer as the *contracted* tensor product (cf. eq. 1) and to the former ones as the *uncontracted* one (cf. eqs. 2,3). Note that reordering factors within a contracted tensor product leads to an equivalent R-matrix because of quasitriangularity.

Within tensor products of arbitrary many representations every single tensor product can be realized as either a contracted or an uncontracted one. After choosing a particular contraction of a tensor product of representations we determine the

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[3] $(\ell_1^+ \otimes \ell_1^+)_n^m = \ell^+{}_k^m \otimes \ell^+{}_n^k$

[4] $\Delta^1 = \Delta, \quad \Delta^n = (\Delta^{n-1} \otimes \mathrm{id}) \circ \Delta$

remaining number of tensor factors s. Now we select one of the 2^s possible universal R-matrices $\mathcal{R}$ of $\mathcal{U}^{\otimes s}$. Pairing it with two copies of the selected contraction gives us an R-matrix for our original (precontracted) representation. (Of course this procedure also works if we take two different representations (with the same s) for the two components of $\mathcal{R}$.)

In the case of the tensor product of three representations we have the following possibilities (neglecting permutations):

$$
\begin{aligned}
s=1 \quad & \langle ([\ ]\times[\ ]\times[\ ]) \otimes ([\ ]\times[\ ]\times[\ ]), \mathcal{R}\rangle, \qquad\qquad \text{here: } \mathcal{R}=\mathcal{R}^{(1)} \\
s=2 \quad & \langle ([\ ]\otimes([\ ]\times[\ ])) \otimes ([\ ]\otimes([\ ]\times[\ ])), \mathcal{R}\rangle \\
& \langle (([\ ]\times[\ ])\otimes[\ ]) \otimes (([\ ]\times[\ ])\otimes[\ ]), \mathcal{R}\rangle \\
s=3 \quad & \langle ([\ ]\otimes[\ ]\otimes[\ ]) \otimes ([\ ]\otimes[\ ]\otimes[\ ]), \mathcal{R}\rangle
\end{aligned}
\tag{24}
$$

Being interested in differential calculi on quantum spaces one has to determine the eigenvalues of $\hat{R} = PR$. As shown by Wess and Zumino [Wess, WeZu] such a differential calculus can be defined if $\hat{R}$ has only one negative eigenvalue (belonging to the antisymmetric projector). The R-matrices arising from our procedure don't have this quality for $s > 2$ as is easily seen by a combinatorial argument. In the case $s = 3$, e.g., we have four contributions $(++-\ ,\ +-+\ ,\ -++\ ,\ ---)$. The eigenvalues depend only on the $\mathcal{R}^{(i)}$ (in eq.(5), and not on $\mathcal{R}^{\mathcal{F}}$ in eq.(8)). To keep the antisymmetric projector from splitting up it is necessary that all three corresponding matrices $\hat{R}^{(i)}$ have exactly two eigenvalues with $\lambda_-^{(i)} = -\lambda_+^{(i)}$. In all cases of interest this is not fulfilled.

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